HISTORY OF MATHEMATICS
MATHEMATICAL TOPIC X
ELLIPSES

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Abstract. Kepler realized that assigning elliptical orbits to the planets greatly simplified the description of their motion. Here we list basic facts about ellipses in modern notation.

1. Ellipses

Definition 1. An *ellipse* is the set of points in a plane such that the sum of the distances from the point to two given points, called *foci*, is a constant, called the *common sum*.

The midpoint between the foci is called the *center*. The line through the foci is called the *major axis*. The line perpendicular to the major axis through the center is called the *minor axis*. The points of intersection of the major axis with the ellipse are called *vertices*. The points of intersection of the minor axis with the ellipse are called *covertices*.

We also call the distance between the vertices the major axis, and half of it is the semimajor axis. Thus the semimajor axis is the distance from the center to a vertex.

2. Kepler's Laws of Planetary Motion

Law 1: The planets move in elliptical orbits with the sun at one vertex.
Law 2: The planets sweep out equal areas in equal amounts of time.
Law 3: The squares of the periods of the planets are proportional to the cubes of their semimajor axes.

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3. Equations

Proposition 1. *Consider the ellipse with foci* $(\pm c, 0)$*, where* $c > 0$*, and common sum* s*. Then the center is* $(0,0)$*, the major axis is* $y = 0$*, the minor axis is* $x = 0$*, the vertices are* $(\pm a, 0)$*, the covertices are* $(0, \pm b)$*, and the equation of the ellipse is*

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,}$$

where

$$\boxed{2a = s} \quad \text{and} \quad \boxed{c^2 = a^2 - b^2}.$$

Proof. The midpoint between the foci is clearly $(0,0)$, so this is the center. Moreover, the line through $(\pm c, 0)$ is the x-axis, so its equation is $y = 0$, and the perpendicular line through the origin is the y-axis, which is $x = 0$.

Suppose that the equation of the ellipse is as stated. If (x, y) is on the intersection of the locus of this equation with the line $y = 0$, then $\frac{x^2}{a^2} = 1$, so $x = \pm a$; thus the vertices are $(\pm a, 0)$. Similarly, the covertices are $(0, \pm b)$.

Now from the definition of an ellipse, the distance from $(a, 0)$ to $(c, 0)$ plus the distance from $(a, 0)$ to $(-c, 0)$ equals s, that is,

$$s = (a - c) + (a + c) = 2a.$$

Moreover, the distance from $(0, b)$ to $(c, 0)$ plus the distance from $(0, b)$ to $(-c, 0)$ equals s. Thus

$$s = \sqrt{(c-0)^2 + (0-b)^2} + \sqrt{(-c-0)^2 + (0-b)^2} = 2\sqrt{c^2 + b^2}.$$

Since $s = 2a$, this gives $a = \sqrt{c^2 + b^2}$, so $a^2 = c^2 + b^2$, which we rewrite as $c^2 = a^2 - b^2$. It remains to derive the equation of the ellipse from the definition.

Let (x, y) be an arbitrary point on the ellipse; from the definition, we have

$$\sqrt{(x-c)^2 + (y-0)^2} + \sqrt{(x-(-c))^2 + (y-0)^2} = s.$$

Subtracting $\sqrt{(x+c)^2 + y^2}$ from both sides and squaring gives

$$(x-c)^2 + y^2 = s^2 + (x+c)^2 + y^2 - 2s\sqrt{(x+c)^2 + y^2}.$$

Rearranging this gives

$$2s\sqrt{(x+c)^2 + y^2} = s^2 + (x+c)^2 - (x-c)^2 = s^2 + 4cx.$$

Dividing by $2s$ and squaring again produces

$$x^2 + 2cx + c^2 + y^2 = \frac{s^2}{4} + 2cx + \frac{4c^2x^2}{s^2}.$$

Cancelling $2cx$ and using that $s^2 = -2a^2$ and $c^2 = a^2 - b^2$ leads us to

$$x^2 + a^2 - b^2 + y^2 = a^2 + \frac{(a^2 - b^2)x^2}{a^2} = a^2 + x^2 - \frac{b^2x^2}{a^2}.$$

Adding $\frac{b^2x^2}{a^2} - x^2 - a^2 + b^2$ to both sides gives

$$\frac{b^2x^2}{a^2} + y^2 = b^2.$$

Finally, dividing by b^2 gives

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

□

4. Eccentricity

The *eccentricity* of an ellipse is

$$\boxed{e = \frac{c}{a} = \frac{\text{distance between foci}}{\text{distance between vertices}}}.$$

Thus $c = ae$.

For an ellipse with $a > b$, we can compute b^2 in terms of a and e as

$$c^2 = a^2 - b^2 \quad \Rightarrow \quad b^2 = a^2 - c^2 = a^2 - a^2e^2 = a^2(1 - e^2).$$

The equation of the ellipse centered at the origin with semimajor axis a and eccentricity e is

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1.}$$

5. Area

Let's use calculus to compute the area of an ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Solving for y gives

$$y = b\sqrt{1 - \frac{x^2}{a^2}} = \frac{b}{a}\sqrt{a^2 - x^2}.$$

Integrating from $-a$ to a gives the area of the upper half of the ellipse:

$$\int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2}\,dx = \frac{b}{a}\int_{-a}^{a}\sqrt{a^2 - x^2}\,dx = \frac{b}{a}\Big[\frac{1}{2}\pi a^2\Big].$$

We recognize the latter integral as that of a a semicircle of radius a, giving the stated value. So the area of the ellipse is double this:

$$\boxed{A = \pi ab}.$$

6. Reflectivity

Proposition 2. *Consider an ellipse with foci F_1 and F_2. Let P be a point on the ellipse and let L_0 be the line through P tangent to the ellipse. Let L_1 be the line through F_1 and P and let L_2 be the line through F_2 and P. Then the angle between L_0 and L_1 equals the angle between L_0 and L_2.*

Remark 1. This says that a wave emitted from one focus bounces off the surface and is transmitted to the other focus.

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